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On the number of moduli of certain algebraic surfaces of general type

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0. Introduction. Let P^3 denote the projective 3-space defined over the field of complex numbers, S an irreducible hypersurface of degree $n=2r$ in P^3, defined by the equation

$$(1) \qquad g^2 + Agh + Bh^2 = 0$$

where g, h, A, and B are homogeneous polynomials of degree r, s, $r-s$, and $2(r-s)$, respectively, with two positive integers $r>s$. Clearly, the curve Δ, defined by $g=h=0$, is contained in the singular locus of S.

We assume that S is generic in the following sense:

1) S has only ordinary singularities (see [4]) and is non-singular outside of Δ.

2) Δ is non-singular.

3) The normalization X of S (which is non-singular by 1)) is a surface of general type.

In [4], Kodaira studied families of surfaces with ordinary singularities in P^3. In particular, he proved that above S belongs to an effectively parametrized family $\mathfrak{F}$ of surfaces S_t, $t \in M_1$, with ordinary singularities in P^3 whose characteristic system on each S_t is complete (see [4], Theorem 8 and § 5.4). In our case, the number $\mu(S)$ of effective parameters of the family $\mathfrak{F}$ is given by

$$\mu(S) = C(r) + C(s) + C(2r-2s) - C(r-2s) - 2$$

where

$$C(m) = \begin{cases} (m+3)(m+2)(m+1)/6 & \text{for } m \geqq 0 \\ 0 & \text{for } m < 0 \end{cases} .$$

On the other hand, Kodaira-Spencer introduced the concept of the number of moduli $m(X)$ of a compact complex manifold X (see [5], Definition 11.1).

Main Theorem. Let S be a generic hypersurface in P^3 defined by the equation (1), X the normalization of S. Then, the number of moduli $m(X)$ is defined, and we have

$$m(X) = \dim H^1(X, \Theta_X) = \mu(S) - 15 - 4\delta_{r,s+1}$$

where Θ_X denotes the sheaf of germs of holomorphic vector fields on X, and $\delta_{r,s+1}$ is Kronecker's delta.

Let $f: X \to P^3$ denote the composition of normalization and the embedding. Then the difference of $\mu(S)$ and

$m(X)$ is the contribution of the number of parameters on which the holomorphic map f depends.

For $(r, s) = (3,1)$ or $(4,3)$, S is one of the examples of M. Noether [6].

For $(r, s) = (3,1)$, X is a minimal algebraic surface with $p_g = 4$, $q = 0$, and $c_1^2 = 6$, where p_g, q, and c_1^2 denote, respectively, the geometric genus, the irregularity, and the Chern number. We have

$$m(X) = 10(p_g - q + 1) - 2c_1^2 = 38,$$

$$H^2(X, \Theta_X) = 0$$

(cf. Kodaira [8]).

For $(r, s) = (4,3)$, X is a complete intersection of two hypersurfaces of degree 2 and 4 in P^4. We have also $H^2(X, \Theta_X) = 0$.

1. Preliminaries. Let E be a hyperplane section of S, $\widetilde{E} = f^*E$, $\widetilde{\Delta} = f^{-1}(\Delta)$. From the equation (1), we infer

Lemma 1. $\widetilde{\Delta}$ <u>is linearly equivalent to</u> $s\widetilde{E}$ <u>on</u> X.

We note that $(n-4)\widetilde{E} - \widetilde{\Delta}$ is a canonical divisor on X, and that

$$H^\nu(X, \mathcal{O}(m\widetilde{E} - \widetilde{\Delta})) \simeq H^\nu(S, \mathcal{O}(mE - \Delta)) \quad \text{for } \nu = 0,1,2$$

(see [4]). By a standard computation (cf. [7]), we get

Lemma 2. $\dim H^0(X, \mathcal{O}(\widetilde{E})) = 4 + \delta_{r,s+1}$, $H^1(X, \mathcal{O}(\widetilde{E})) = 0$.

Lemma 3. 1) <u>The canonical bundle</u> K <u>of</u> X <u>is ample</u>. <u>In particular</u>, X <u>is minimal</u>.

2)
$$p_g = \binom{2r-1}{3} - (1/2)rs(3r-s-4),$$
$$q = 0,$$
$$c_1^2 = 2r(2r-s-4)^2.$$

Remark. If $r = s+1$, then the <u>complete</u> linear system $|\tilde{E}|$ is very ample, and X is a complete intersection of two hypersurfaces of degree 2 and r in P^4.

2. Relation between deformations of S and X. Let $\{S_t\}_{t\in M}$ be a family of surfaces of degree n in P^3 with ordinary singularities, $S = S_0$, $0 \in M$. Letting $T_0(M)$ denote the tangent space of M at 0, we have the characteristic map

$$\sigma : T_0(M) \longrightarrow H^0(S, \mathcal{N}_S^0)$$

where $\mathcal{N}_S^0$ denotes the sheaf $\mathcal{O}(nE - \Delta - \Sigma \tau_i')$ in the notation of [4]. We note that we have an exact sequence

$$0 \longrightarrow \Theta_S \longrightarrow \Theta_{P^3|S} \longrightarrow \mathcal{N}_S^0 \longrightarrow 0 .$$

On the other hand, the normalization X_t of S_t form a family $\mathfrak{X} = \{X_t\}_{t\in M}$ of deformations of $X = X_0$ and the holomorphic map $f : X \longrightarrow P^3$ extends to a holomorphic map $\Phi : \mathfrak{X} \longrightarrow P^3 \times M$ over M. Let Θ_X and Θ_{P^3} denote the sheaves of germs of holomorphic vector fields on X and P^3 respectively, and let $\mathcal{J}_{X/P^3}$ denote the cokernel of the canonical homomorphism $F: \Theta_X \longrightarrow f^*\Theta_{P^3}$. Then we have the characteristic map

$$\tau : T_0(M) \longrightarrow H^0(X, \mathcal{J}_{X/P^3})$$

(see [3], §1).

Lemma 4. <u>There is a canonical isomorphism</u>

$$\mathfrak{f} : \mathcal{N}_S^0 \longrightarrow f_*\mathcal{J}_{X/P^3}$$

<u>which induces an isomorphism</u>

$$\mathfrak{f} : H^0(S, \mathcal{N}_S^0) \longrightarrow H^0(X, \mathcal{J}_{X/P^3})$$

<u>such that</u> $-\tau = \mathfrak{f}\circ\sigma$.

We have two exact sequences

$$0 \longrightarrow \Theta_S \longrightarrow \Theta_{P^3}|_S \longrightarrow \mathcal{N}_S^0 \longrightarrow 0 ,$$

$$0 \longrightarrow f_*\Theta_X \rightarrow f_*f^*\Theta_{P^3} \rightarrow f_*\mathcal{J}_{X/P^3} \rightarrow 0 .$$

Moreover, there exists a canonical homomorphism

$$f^*: \Theta_{P^3}|_S \longrightarrow f_*f^*\Theta_{P^3} .$$

One can easily see that f^* induces a disired isomorphism.

3. Vanishing of obstructions.

Lemma 5. <u>The coboundary map</u>

$$\delta : H^0(X, \mathcal{J}_{X/P^3}) \longrightarrow H^1(X, \Theta_X)$$

is surjective.

Proof. Let $\rho \in H^1(X, \Theta_X)$. Then ρ corresponds to a deformation X_ρ of X over $I = \mathrm{Spec}\ \mathbb{C}[t]/(t^2)$.

By Lemma 1, we have $K = (n-s-4)[\tilde{E}]$. It follows that the line bundle $[\tilde{E}]$ extends to a line bundle on X_ρ. Then, by Lemma 2, the holomorphic map f extends to a holomorphic map $X_\rho \to P^3 \times I$ over I. This means that δ is surjective.

By the result of Kodaira cited in Introduction, and by Lemma 4, we obtain a family $\mathfrak{X}_1 = \{X_t\}_{t \in M_1}$ of deformations of $X = X_0$ such that

$$\tau : T_0(M_1) \longrightarrow H^0(X, \mathcal{S}_{X/P^3})$$

is surjective. By Lemma 5 and [3], Proposition 1.4, the infinitesimal deformation map

$$\rho : T_0(M_1) \longrightarrow H^1(X, \Theta_X)$$

is surjective.

This implies the existence of an effectively parametrized complete family of deformations of X and the equality $m(X) = \dim H^1(X, \Theta_X)$.

On the other hand, we have

$$\dim H^0(X, f^*\Theta_{P^3}) = 15 + 4\delta_{r,s+1}$$

by Lemmas 2 and 3. Finally it follows that

$$\dim H^1(X, \Theta_X) = \mu(S) - 15 - 4\,\delta_{r,s+1}$$

by Lemma 5.

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